Non-Uniform Flow

1. (F 6.10)

Cancelled.

2. (F 6.13)

(Please note: $z = 1.5$ is used in the following solution, but $z = 2$ is given in the problem).

Determine normal flow depth. The Manning formula is written:

$$Q = \frac{1}{n} AR^{2/3} S_o^{1/2}$$

The flow area, the wetted perimeter, and the hydraulic radius is determined to be (following the formulas in French page 12):

$$A = (9.1 + 1.5y_N)y_N$$

$$P = 9.1 + 2y_N\sqrt{1+1.5^2}$$

$$R = \frac{A}{P} = \frac{(9.1 + 1.5y_N)y_N}{9.1 + 2y_N\sqrt{3.25}}$$

Enter $A$ and $R$ into the Manning equation:

$$71 = \frac{1}{0.027} (9.1 + 1.5y_N)y_N \left(\frac{(9.1 + 1.5y_N)y_N}{9.1 + 2y_N\sqrt{3.25}}\right)^{2/3} 0.001^{1/2}$$

$$60.66 = \left(\frac{(9.1 + 1.5y_N)y_N^{5/3}}{(9.1 + 3.6y_N)^{2/3}}\right)$$

$\rightarrow y_N = 2.85$ m (trial and error)
Check the flow conditions in the channel by calculating the Froude number. First, the mean velocity is given by:

\[ u_N = \frac{Q}{A} = \frac{71}{(9.1 + 1.5 \cdot 2.85) \cdot 2.85} = 1.86 \text{ m/s} \]

The Froude number is obtained from:

\[ Fr = \frac{u_N}{\sqrt{gA/T}} = \frac{1.86\sqrt{9.1 + 1.5 \cdot 2.85 \cdot 2}}{\sqrt{9.81 \cdot (9.1 + 1.5 \cdot 2.85) \cdot 2.85}} = 0.35 < 1 \]

Since \( Fr < 1 \), the flow is subcritical far upstream in the channel where the lake will not influence the flow.

Determine the critical depth:

\[ Fr = \frac{u_c}{\sqrt{gA_c / T_c}} = \frac{Q}{A_c \sqrt{gA_c / T_c}} = 1 \]

\[ \rightarrow \quad \frac{Q^2}{g} = \frac{A_c^3}{T_c} \]

\[ \frac{71^2}{g} = \frac{A_c^3}{T_c} = \left( (9.1 + 1.5y_c) y_c \right)^3 / 9.1 + 3y_c \]

Trial and error gives \( y_c = 1.67 \text{ m} \)

Again, since \( y_N > y_c \) the flow is subcritical far upstream in the channel.

Some distance up in the channel uniform flow conditions will prevail (the channel is regarded as long), but as the flow approaches the lake two scenarios are possible: (1) the lake level is low, not influencing the channel flow, and a critical section is found at the downstream end of the channel; and (2) the lake level is sufficiently high to cause backwater effects into the channel. In both cases a stretch of non-uniform flow develops upstream the lake, but in the former case the water depth at the downstream end of the channel is equal to \( y_c \), whereas in the latter case the lake water level determines the downstream water depth.

The detailed shape of the water surface profile upstream the lake is determined through a step calculation, starting at the control point and proceeding upstream, against the flow (subcritical flow conditions).
The energy equation between two locations \( i \) and \( i+1 \), including friction, is given by,

\[
y_i + \frac{\alpha u_i^2}{2g} + z_i = y_{i+1} + \frac{\alpha u_{i+1}^2}{2g} + z_{i+1} + h_{li,i+1}
\]

where \( h_{li,i+1} \) is the energy loss between \( i \) and \( i+1 \). Introducing the slope of the channel \( (S_o) \) and energy grade line \( (S_f) \) into the energy equation gives:

\[
y_i + \frac{\alpha u_i^2}{2g} - y_{i+1} - \frac{\alpha u_{i+1}^2}{2g} = S_f \Delta x - S_o \Delta x
\]

\[
\Delta x = \frac{y_i + \frac{\alpha u_i^2}{2g} - y_{i+1} - \frac{\alpha u_{i+1}^2}{2g}}{S_f - S_o}
\]

The mean energy loss is estimated from:

\[
S_f = \frac{1}{2} \left( S_{fi} + S_{fi+1} \right)
\]

\[
S_{fi} = \frac{n^2 u^2}{R^{4/3}}
\]

Make a table to perform the step calculation, starting at the appropriate water depth and move upstream the distance of interest. Use suitable steps in the depth.

a.

The lake water level is 4.3 m above datum and the invert of the channel is 3.0 above datum at the most downstream end of the channel. Thus, the lake water level above the invert reaches to \( 4.3 - 3.0 = 1.3 \) m < \( y_c = 1.67 \) m. Thus, a critical section will form in this case (no backwater effects).

Make a table and perform the step calculations, starting at 1.67 m.
And so on until the required distance has been attained.

3. (A1)

This is a case of spatially varied flow, that is, $Q$ increases in the downstream direction as water flows in laterally to the initial part of the channel.

The total flow rate at the downstream end of the initial, horizontal part of the channel, where the lateral inflow occurs, is ($x = 10$ m):

$$Q = q_{in} x_{10} = 0.063 \cdot 10 = 0.63 \text{ m}^3/\text{s}$$

Assuming that the channel is long downstream $x = 10$ m and that the bottom slope is mild, normal water depth exist at $x = 10$ m.

Determine normal water depth. The Manning formula is written:
\[ Q = \frac{1}{n} AR^{2/3} S_o^{1/2} \]

\[
0.63 = \frac{1}{0.017} y_N \cdot 1 \left( \frac{y_N \cdot 1}{2 y_N + 1} \right)^{2/3} 0.0005^{1/2} 
\]

\[ \rightarrow \quad y_N = 1.0 \text{ m} \]

Check the Froude number to see that the assumption about having normal water depth at the outflow point is correct:

\[
Fr = \frac{u_N}{\sqrt{g y_N}} = \frac{0.63 / (1.1)}{\sqrt{9.81 \cdot 1}} = 0.20 < 1
\]

Thus, the flow is subcritical in the channel and the assumption made is correct.

To calculate the water depth at the upstream end \( y_{st} (x = 0) \) the momentum equation is employed for a control volume encompassing the entire horizontal part of the channel (\( x = 0 \) to \( x = 10 \) m) yielding:

\[
\rho g \frac{y_{st}^2}{2} b - \rho g \frac{y_{10}^2}{2} b = \rho Q_{10} u_{10} - \rho Q_{0} u_{0}
\]

\[
\rho g \frac{y_{st}^2}{2} 1 - \rho g \frac{1^2}{2} 1 = \rho 0.63 \cdot 1.1 - \rho \cdot 0
\]

\[ \rightarrow \quad y_{st} = 1.04 \text{ m} \]

To determine the entire water surface profile, calculate \( y \) at given \( x \) with a control volume going from \( x = 0 \) to \( x \). Thus, the momentum equation becomes,

\[
\rho g \frac{y_{st}^2}{2} b - \rho g \frac{y^2}{2} b = \rho Q_s u_x
\]

where:

\[ Q_s = q_s x \]

\[ u_x = \frac{Q_s}{y b} \]
The following equation is obtained:

\[ \rho g \frac{y_{st}^2}{2} b - \rho g \frac{y^2}{2} b = \rho \frac{(q_{in} x)^2}{by} \]

This equation may be written:

\[ y \left( y_{st}^2 - y^2 \right) = 2 \frac{(q_{in} x)^2}{gb^2} \]

The result is a third-degree polynomial that has to be solved through some numerical procedure.

An alternative approach to obtain the water surface profile is to solve the governing equation (see French page 249; neglect friction and bottom slope):

\[ \frac{dQ}{dx} = -\frac{2Q}{gA^2} \frac{dQ}{dx} - \frac{Q^2}{gA^2 D} \]

Introducing \( Q = q_{in} x \), yielding \( dQ/dx = q_{in} \), and \( D = A/T = y \) gives:

\[ \frac{dy}{dx} = -\frac{2q_{in}^2 x}{g y^3} \approx -\frac{2q_{in}^2 x}{g y^2} \]

The second term in the denominator can be shown to be small. Integrating this equation with the boundary condition \( x = 0, y = y_{st} \) yields:

\[ y = \left( y_{st}^3 - \frac{3q_{in}^2}{g} x^2 \right)^{1/3} \]